COUPLED PAINLEVÉ III SYSTEM WITH AFFINE WEYL GROUP SYMMETRY OF TYPE $D_6^{(1)}$

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Abstract. We find and study a six-parameter family of coupled Painlevé III systems in dimension six with affine Weyl group symmetry of type $D_6^{(1)}$. We also find and study its degenerate systems with affine Weyl group symmetry of types $B_5^{(1)}$ and $D_5^{(2)}$.

1. Introduction

In [13, 14], we presented a 4-parameter family of 2-coupled Painlevé III systems in dimension four with affine Weyl group symmetry of type $D_4^{(1)}$. We will make non-linear ordinary differential systems with affine Weyl group symmetry of type $D_{2n+2}^{(1)}$ $(n \geq 2)$.

In [10, 12], we succeeded to make $(2n+2)$-parameter family of n-coupled Painlevé VI systems in dimension 2n with affine Weyl group symmetry of type $D_{2n+2}^{(1)}$ $(n \geq 1)$ by connecting the invariant divisors $p_i, q_i - q_{i+1}, p_{i+1}$ for the canonical variables (q_i, p_i) $(i = 1, 2, \dots, n)$. These systems are polynomial Hamiltonian systems with coupled Painlevé VI Hamiltonians given by

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \quad (i = 1, 2, \dots, n) \tag{1}$$

with the polynomial Hamiltonian

$$H = \sum_{i=1}^{n} H_{VI}(q_i, p_i, t; \alpha_0^{(i)}, \alpha_1^{(i)}, \dots, \alpha_4^{(i)}) + \sum_{1 \leqslant l < m \leqslant n} \frac{2(q_l - t)p_l q_m((q_m - 1)p_m + \alpha_2^{(m)})}{t(t-1)}, \tag{2}$$

where the symbol $H_{VI}(x, y, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is given by

$$\begin{aligned} &H_{VI}(x, y, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) \\ &= \frac{1}{t(t-1)}[y^2(x-t)(x-1)x - \{(\alpha_0 - 1)(x-1)x + \alpha_3(x-t)x \\ &+ \alpha_4(x-t)(x-1)\}y + \alpha_2(\alpha_1 + \alpha_2)x] \quad (\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1). \end{aligned} \tag{3}$$

However, in this case the iniariant divisors are different from the ones of P_{VI}-case.

Invariant divisors	f_0	f_1	f_2	f_3	f_4
P_{VI}	$q - t$	$q - \infty$	p	$q - 1$	q
2-CPIII	$p_1 - 1$	p_1	$q_1 q_2 - 1$	p_2	$p_2 - t$

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At first, let us consider $D_6^{(1)}$ case. In this paper, we present a 6-parameter family of coupled Painlevé III systems with affine Weyl group symmetry of type $D_6^{(1)}$ explicitly given by

$$\frac{dx}{dt}=\frac{\partial H}{\partial y},\quad \frac{dy}{dt}=-\frac{\partial H}{\partial x},\quad \frac{dz}{dt}=\frac{\partial H}{\partial w},\quad \frac{dw}{dt}=-\frac{\partial H}{\partial z},\quad \frac{dq}{dt}=\frac{\partial H}{\partial p},\quad \frac{dp}{dt}=-\frac{\partial H}{\partial q} \tag{4}$$

with the polynomial Hamiltonian

$$\begin{aligned} H=&\frac{x^2(y-1)y+x\{(\alpha_0+\alpha_1)y-\alpha_1\}+ty}{t}\\ &+\frac{z^2(w-1)w+z\{(\alpha_0+\alpha_1+2\alpha_2+2\alpha_3)w-\alpha_3\}+tw}{t}\\ &+\frac{q^2(p-t)p+q\{(\alpha_5+\alpha_6-1)p-t\alpha_5\}+p}{t}+\frac{2yz(zw+\alpha_3)}{t}-\frac{2(y+w)p}{t}\\ &=H_{III}(x,y,t;\alpha_1,\alpha_0)+H_{III}(z,w,t;\alpha_3,\alpha_0+\alpha_1+2\alpha_2+\alpha_3)\\ &+\tilde{H}_{III}(q,p,t;\alpha_5,1-\alpha_6)+\frac{2yz(zw+\alpha_3)}{t}-\frac{2(y+w)p}{t}. \end{aligned} \tag{5}$$

Here x,y,z,w,q and p denote unknown complex variables, and $\alpha_0,\alpha_1,\dots,\alpha_6$ are complex parameters satisfying the relation:

$$\alpha_0+\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6=1. \tag{6}$$

The symbols $H_{III},\tilde{H}_{III}$ are given by

$$H_{III}(u,v,t;\gamma_0,\gamma_1,\gamma_2)=\frac{u^2v(v-1)+u\{(\gamma_0+\gamma_2)v-\gamma_0\}+tv}{t}\quad (\gamma_0+2\gamma_1+\gamma_2=1), \tag{7}$$

$$\tilde{H}_{III}(U,V,t;\gamma_0,\gamma_1,\gamma_2)=\frac{U^2V(V-t)-U\{(-\gamma_0+\gamma_2)V+\gamma_0t\}+V}{t}. \tag{8}$$

The relation between (u,v) and (U,V) is given by

$$(U,V)=(1/u,-u(vu+\gamma_0)). \tag{9}$$

We remark that for this system we tried to seek its first integrals of polynomial type with respect to x,y,z,w,q,p. However, we can not find. Of course, the Hamiltonian H is not the first integral.

This is the second example which gave higher order Painlevé type systems of type $D_6^{(1)}$.

We also find and study its degenerate systems with affine Weyl group symmetry of types $B_5^{(1)}$ and $D_5^{(2)}$. In $D_5^{(2)}$-case, each differential system with respect to all principal parts has its first integral. Nevertheless, the polynomial Hamiltonian itself is not its first integral (see Section 4).

We give an explicit confluence process from the $D_6^{(1)}$ system, respectively. We will show that the system of type $B_5^{(1)}$ is equivalent to the $D_5^{(1)}$ system (see [15]) by explicit birational and symplectic transformations with some parameter's changes (see Section 3).

2. The system of type $D_6^{(1)}$

In this section, we present a 6-parameter family of coupled Painlevé III systems with affine Weyl group symmetry of type $D_6^{(1)}$ explicitly given by

$$
(10) \qquad \begin{cases}
\dfrac{dx}{dt} = \dfrac{\partial H}{\partial y} = \dfrac{2x^2y + 2z^2w - x^2 - (\alpha_0+\alpha_1)x + 2\alpha_3 z - 2p + t}{t}, \\
\dfrac{dy}{dt} = -\dfrac{\partial H}{\partial x} = \dfrac{-2xy^2 + 2xy - (\alpha_0+\alpha_1)y + \alpha_1}{t}, \\
\dfrac{dz}{dt} = \dfrac{\partial H}{\partial w} = \dfrac{2z^2w + 2yz^2 - z^2 - (2\alpha_4 - 1 + \alpha_5 + \alpha_6)z - 2p + t}{t}, \\
\dfrac{dw}{dt} = -\dfrac{\partial H}{\partial z} = \dfrac{-2zw^2 - 4yzw + 2zw - 2\alpha_3 y + (2\alpha_4 - 1 + \alpha_5 + \alpha_6)w + \alpha_3}{t}, \\
\dfrac{dq}{dt} = \dfrac{\partial H}{\partial p} = \dfrac{2q^2p - tq^2 - 2y - 2w + (\alpha_5+\alpha_6-1)q + 1}{t}, \\
\dfrac{dp}{dt} = -\dfrac{\partial H}{\partial q} = \dfrac{-2qp^2 + 2tqp - (\alpha_5+\alpha_6-1)p + t\alpha_5}{t}
\end{cases}
$$

with the Hamiltonian (5).

Theorem 2.1. *The system* (10) *admits extended affine Weyl group symmetry of type $D_6^{(1)}$ as the group of its Bäcklund transformations whose generators are explicitly given as follows*: *with the notation* $(*) := (x, y, z, w, q, p, t; \alpha_0, \alpha_1, \dots, \alpha_6)$,

$$s_0 : (*) \to \left(x + \frac{\alpha_0}{y-1}, y, z, w, q, p, t; -\alpha_0, \alpha_1, \alpha_2 + \alpha_0, \alpha_3, \alpha_4, \alpha_5, \alpha_6\right),$$

$$s_1 : (*) \to \left(x + \frac{\alpha_1}{y}, y, z, w, q, p, t; \alpha_0, -\alpha_1, \alpha_2 + \alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\right),$$

$$s_2 : (*) \to \left(x, y - \frac{\alpha_2}{x-z}, z, w + \frac{\alpha_2}{x-z}, q, p, t; \alpha_0 + \alpha_2, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4, \alpha_5, \alpha_6\right),$$

$$s_3 : (*) \to \left(x, y, z + \frac{\alpha_3}{w}, w, q, p, t; \alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4 + \alpha_3, \alpha_5, \alpha_6\right),$$

$$s_4 : (*) \to \left(x, y, z, w - \frac{\alpha_4 q}{zq-1}, q, p - \frac{\alpha_4 z}{zq-1}, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3 + \alpha_4, -\alpha_4, \alpha_5 + \alpha_4, \alpha_6 + \alpha_4\right),$$

$$s_5 : (*) \to \left(x, y, z, w, q + \frac{\alpha_5}{p}, p, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4 + \alpha_5, -\alpha_5, \alpha_6\right),$$

$$s_6 : (*) \to \left(x, y, z, w, q + \frac{\alpha_6}{p-t}, p, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4 + \alpha_6, \alpha_5, -\alpha_6\right),$$

$$\pi_1 : (*) \to (-x, 1-y, -z, -w, -q, t-p, t; \alpha_1, \alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_6, \alpha_5),$$

$$\pi_2 : (*) \to \left(tq, \frac{p}{t}, \frac{t}{z}, -\frac{(zw+\alpha_3)z}{t}, \frac{x}{t}, ty, t; \alpha_6, \alpha_5, \alpha_4, \alpha_3, \alpha_2, \alpha_1, \alpha_0\right),$$

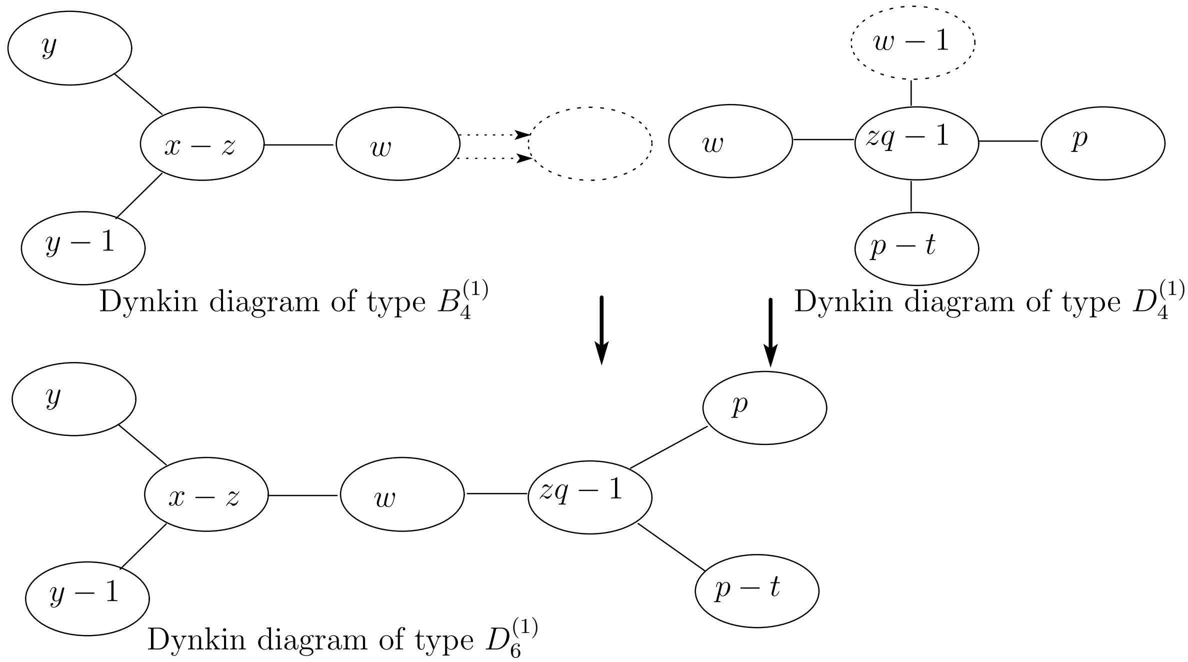

FIGURE 1. The figure denotes the Dynkin diagram of types $B_4^{(1)}, D_4^{(1)}$ and $D_6^{(1)}$. The symbol in each circle denotes the invariant divisors of the systems of types $B_4^{(1)}, D_4^{(1)}$ and $D_6^{(1)}$.

$$\pi_3 : (*) \rightarrow (x, y, z, w, q, p-t, -t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6, \alpha_5),$$
$$\pi_4 : (*) \rightarrow (-x, 1-y, -z, -w, -q, -p, -t; \alpha_1, \alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6).$$

The Bäcklund transformations of this system satisfy the universal description for $D_6^{(1)}$ root system. Since these universal Bäcklund transformations have Lie theoretic origin, similarity reduction of a Drinfeld-Sokolov hierarchy admits such a Bäcklund symmetry.

PROPOSITION 2.2. *Let us define the following translation operators*:

$$\begin{aligned} &T_1 := \pi_1 s_5 s_4 s_3 s_2 s_1 s_0 s_1 s_2 s_3 s_4 s_5, \quad T_2 := s_4 s_6 T_1 s_6 s_4, \\ &T_3 := s_6 T_1 s_6, \quad T_4 := \pi_2 T_1 \pi_2, \quad T_5 := \pi_2 T_2 \pi_2, \quad T_6 := \pi_2 T_3 \pi_2. \end{aligned} \tag{11}$$

These translation operators act on parameters α_i *as follows*:

$$\begin{aligned} T_1(\alpha_0, \alpha_1, \dots, \alpha_6) =&(\alpha_0, \alpha_1, \dots, \alpha_6) + (0, 0, 0, 0, 0, -1, 1), \\ T_2(\alpha_0, \alpha_1, \dots, \alpha_6) =&(\alpha_0, \alpha_1, \dots, \alpha_6) + (0, 0, 0, 1, -1, 0, 0), \\ T_3(\alpha_0, \alpha_1, \dots, \alpha_6) =&(\alpha_0, \alpha_1, \dots, \alpha_6) + (0, 0, 0, 0, 1, -1, -1), \\ T_4(\alpha_0, \alpha_1, \dots, \alpha_6) =&(\alpha_0, \alpha_1, \dots, \alpha_6) + (1, -1, 0, 0, 0, 0, 0), \\ T_5(\alpha_0, \alpha_1, \dots, \alpha_6) =&(\alpha_0, \alpha_1, \dots, \alpha_6) + (0, 0, -1, 1, 0, 0, 0), \\ T_6(\alpha_0, \alpha_1, \dots, \alpha_6) =&(\alpha_0, \alpha_1, \dots, \alpha_6) + (-1, -1, 1, 0, 0, 0, 0). \end{aligned} \tag{12}$$

THEOREM 2.3. *Let us consider a polynomial Hamiltonian system with Hamiltonian* $H \in \mathbb{C}(t)[x, y, z, w, q, p]$. *We assume that*

(A1) $deg(H)=4$ *with respect to* x,y,z,w,q,p.

(A2) This system becomes again a polynomial Hamiltonian system in each coordinate system r_i $(i=0,1,2,3,5,6)$:

$$\begin{aligned}
&r_0: x_0=1/x,\ y_0=-((y-1)x+\alpha_0)x,\ z_0=z,\ w_0=w,\ q_0=q,\ p_0=p,\\
&r_1: x_1=1/x,\ y_1=-(yx+\alpha_1)x,\ z_1=z,\ w_1=w,\ q_1=q,\ p_1=p,\\
&r_2: x_2=-((x-z)y-\alpha_2)y,\ y_2=1/y,\ z_2=z,\ w_2=w+y,\ q_2=q,\ p_2=p,\\
&r_3: x_3=x,\ y_3=y,\ z_3=1/z,\ w_3=-(wz+\alpha_3)z,\ q_3=q,\ p_3=p,\\
&r_5: x_5=x,\ y_5=y,\ z_5=z,\ w_5=w,\ q_5=1/q,\ p_5=-(pq+\alpha_5)q,\\
&r_6: x_6=x,\ y_6=y,\ z_6=z,\ w_6=w.\ q_6=1/q,\ p_6=-((p-t)q+\alpha_6)q.
\end{aligned}$$

(A3) In addition to the assumption (A2), the Hamiltonian system in the coordinate system r_3 *becomes again a polynomial Hamiltonian system in the coordinate system* r_4:

$$r_4: x_4=x_3,\ y_4=y_3,\ z_4=-((z_3-q_3)w_3-\alpha_4)w_3,\ w_4=1/w_3,\ q_4=q_3,\ p_4=p_3+w_3.$$

Then such a system coincides with the system (10) *with the polynomial Hamiltonian* (5).

We note that the conditions $(A2)$ and $(A3)$ should be read that

$$r_j(H)\quad(j=0,1,2,3,5),\quad r_6(H+q),\quad r_4(r_3(H))$$

are polynomials with respect to x,y,z,w,q,p or x_3,y_3,z_3,w_3,q_3,p_3.

Finally, we consider the rational and algebraic solutions of the system (10).

At first, we consider the Dynkin diagram automorphism π_1. By this transformation, the fixed solution is derived from

$$\begin{aligned}
&\alpha_0=\alpha_1,\quad \alpha_1=\alpha_0,\quad \alpha_5=\alpha_6,\quad \alpha_6=\alpha_5,\\
&x=-x,\quad y=1-y,\quad z=-z,\quad w=-w,\quad q=-q,\quad p=t-p.
\end{aligned}\tag{13}$$

Then we obtain

$$\begin{aligned}
&(\alpha_0,\alpha_1,\dots,\alpha_6)=\left(\frac{1}{2}-\alpha_2-\alpha_3-\alpha_4-\alpha_6,\frac{1}{2}-\alpha_2-\alpha_3-\alpha_4-\alpha_6,\alpha_2,\alpha_3,\alpha_4,\alpha_6,\alpha_6\right),\\
&(x,y,z,w,q,p)=\left(0,\frac{1}{2},0,0,0,\frac{t}{2}\right).
\end{aligned}\tag{14}$$

Next, we find two algebraic solutions:

$$\begin{aligned}
&(\alpha_0,\alpha_1,\dots,\alpha_6)=\left(\frac{1-2\alpha_3}{2},0,0,\alpha_3,0,0,\frac{1-2\alpha_3}{2}\right),\\
&(x,y,z,w,q,p)=\left(\sqrt{t},0,\sqrt{t},-\frac{\alpha_3}{2\sqrt{t}},\frac{1}{\sqrt{t}},0\right)
\end{aligned}\tag{15}$$

and

$$(16)\quad \begin{aligned} &(\alpha_0,\alpha_1,\dots,\alpha_6)=\left(0,\frac{1+2\alpha_3}{2},-\alpha_3,\alpha_3,-\alpha_3,\frac{1+2\alpha_3}{2},0\right),\\ &(x,y,z,w,q,p)=\left(-\sqrt{t},1,\sqrt{t},-\frac{\alpha_3}{2\sqrt{t}},-\frac{1}{\sqrt{t}},t\right). \end{aligned}$$

3. The system of type $B_5^{(1)}$

In this section, we present a 5-parameter family of coupled Painlevé systems with affine Weyl group symmetry of type $B_5^{(1)}$ explicitly given by

$$(17)\quad \begin{cases} \dfrac{dx}{dt}=\dfrac{\partial H}{\partial y}=\dfrac{2x^2y+2z^2w+2\alpha_0x+2\alpha_2z-2p+t}{t},\\ \dfrac{dy}{dt}=-\dfrac{\partial H}{\partial x}=\dfrac{-2xy^2-2\alpha_0y-1}{t},\\ \dfrac{dz}{dt}=\dfrac{\partial H}{\partial w}=\dfrac{2z^2w+2yz^2+2(\alpha_0+\alpha_1+\alpha_2)z-2p+t}{t},\\ \dfrac{dw}{dt}=-\dfrac{\partial H}{\partial z}=\dfrac{-2zw^2-4yzw-2\alpha_2y-2(\alpha_0+\alpha_1+\alpha_2)w}{t},\\ \dfrac{dq}{dt}=\dfrac{\partial H}{\partial p}=\dfrac{2q^2p-tq^2-2y-2w-2(\alpha_0+\alpha_1+\alpha_2+\alpha_3)q}{t},\\ \dfrac{dp}{dt}=-\dfrac{\partial H}{\partial q}=\dfrac{-2qp^2+2tqp+2(\alpha_0+\alpha_1+\alpha_2+\alpha_3)p+t\alpha_4}{t} \end{cases}$$

with the polynomial Hamiltonian

$$(18)\quad \begin{aligned} H=&\frac{x^2y^2+2\alpha_0xy+x+ty}{t}+\frac{z^2w^2+2(\alpha_0+\alpha_1+\alpha_2)zw+tw}{t}\\ &+\frac{q^2p^2-tq^2p+(\alpha_4+\alpha_5-1)qp-\alpha_4tq}{t}+\frac{2yz(zw+\alpha_2)}{t}-\frac{2(y+w)p}{t}\\ =&H_{III}^{D_7^{(1)}}(x,y,t;2\alpha_0)+H_1(z,w,t;2(\alpha_0+\alpha_1+\alpha_2))\\ &+H_2(q,p,t;\alpha_4+\alpha_5-1,-\alpha_4)+\frac{2yz(zw+\alpha_2)}{t}-\frac{2(y+w)p}{t}. \end{aligned}$$

Here x,y,z,w,q and p denote unknown complex variables, and $\alpha_0,\alpha_1,\dots,\alpha_5$ are complex parameters satisfying the relation:

$$(19)\quad 2\alpha_0+2\alpha_1+2\alpha_2+2\alpha_3+\alpha_4+\alpha_5=1.$$

The symbols $H_{III}^{D_7^{(1)}}$, H_1 and H_2 are given by

$$H_{III}^{D_7^{(1)}}(q,p,t;\beta_1) = \frac{q^2p^2+\beta_1 qp+q+tp}{t} \quad (\beta_0+\beta_1=1), \tag{20}$$

$$H_1(q,p,t;\alpha) = \frac{q^2p^2+\alpha qp+tp}{t}, \tag{21}$$

$$H_2(q,p,t;\alpha,\beta) = \frac{q^2p^2-tq^2p+\alpha qp+\beta tq}{t}. \tag{22}$$

We remark that for $y=q/\tau,\ t=\tau^2$ the Hamiltonian system with $H_{III}^{D_7^{(1)}}$ is the special case of the third Painlevé system (see [17]):

$$\frac{d^2y}{d\tau^2} = \frac{1}{y}\left(\frac{dy}{d\tau}\right)^2 - \frac{1}{\tau}\frac{dy}{d\tau} + \frac{1}{\tau}(ay^2+b) + cy^3 + \frac{d}{y} \tag{23}$$

with

$$a=-8, \quad b=4(1-\beta_1), \quad c=0, \quad d=-4. \tag{24}$$

From the viewpoint of symmetry, the Hamiltonian system

$$\frac{dq}{dt} = \frac{\partial H_{III}^{D_7^{(1)}}}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H_{III}^{D_7^{(1)}}}{\partial q} \tag{25}$$

has extended affine Weyl group symmetry of type $A_1^{(1)}$, whose generators $< s_0, s_1, \pi = \sigma \circ s_1 >$ are explicitly given as follows (see [17]):

$$\begin{cases} s_0(q,p,t;\beta_0,\beta_1) = \left(q, p+\dfrac{\beta_0}{q}-\dfrac{t}{q^2}, -t; -\beta_0, \beta_1+2\beta_0\right), \\ s_1(q,p,t;\beta_0,\beta_1) = \left(-q+\dfrac{\beta_1}{p}+\dfrac{1}{p^2}, -p, -t; \beta_0+2\beta_1, -\beta_1\right), \\ \sigma(q,p,t;\beta_0,\beta_1) = \left(tp, -\dfrac{q}{t}, -t; \beta_1, \beta_0\right). \end{cases} \tag{26}$$

PROPOSITION 3.1. *By the following birational and symplectic transformations* tr_i $(i = 1,2)$:

$$\begin{cases} tr_1(q,p) = (q/t, tp), \\ tr_2(q,p) = (-p/t, tq), \end{cases} \tag{27}$$

the Hamiltonians H_1 *and* H_2 *satisfy the following relations*:

$$tr_1(H_1) = \frac{q^2p^2+(\alpha-1)qp+p}{t}, \quad tr_2(H_2) = \frac{q^2p^2+qp^2-(\alpha+1)qp+\beta p}{t}. \tag{28}$$

Here, for notational convenience, we use the same symbol q, p, α.

By Proposition 3.1, we see that the Hamiltonian system with

$$K_1 := \frac{q^2p^2+(\alpha-1)qp+p}{t}$$

has the first integral.

Proposition 3.2. *The system with the Hamiltonian* K_1

$$\frac{dq}{dt}=\frac{\partial K_1}{\partial p}, \quad \frac{dp}{dt}=-\frac{\partial K_1}{\partial q} \tag{29}$$

has the first integral I_1:

$$I_1=q^2p^2+(\alpha-1)qp+p. \tag{30}$$

We see that the relation between the Hamiltonian K_1 and the first integral I_1 is explicitly given by

$$tK_1=I_1. \tag{31}$$

We also show that the Hamiltonian system with

$$K_2:=\frac{q^2p^2+qp^2-(\alpha+1)qp+\beta p}{t}$$

has the first integral.

Proposition 3.3. *The system with the Hamiltonian* K_2

$$\frac{dq}{dt}=\frac{\partial K_2}{\partial p}, \quad \frac{dp}{dt}=-\frac{\partial K_2}{\partial q} \tag{32}$$

has the first integral I_2:

$$I_2=q^2p^2+qp^2-(\alpha+1)qp+\beta p. \tag{33}$$

We see that the relation between the Hamiltonian K_2 and the first integral I_2 is explicitly given by

$$tK_2=I_2. \tag{34}$$

We remark that for this system we tried to seek its first integrals of polynomial type with respect to x,y,z,w,q,p. However, we can not find. Of course, the Hamiltonian H is not the first integral.

Theorem 3.4. *The system* (17) *admits extended affine Weyl group symmetry of type* $B_5^{(1)}$ *as the group of its Bäcklund transformations, whose generators are explicitly given*

as follows: with the notation $(*) := (x, y, z, w, q, p, t; \alpha_0, \alpha_1, \dots, \alpha_5)$,

$$
\begin{aligned}
s_0 &: (*) \to \left(-x - \frac{2\alpha_0}{y} - \frac{1}{y^2}, -y, -z, -w, -q, -p, -t; -\alpha_0, \alpha_1 + 2\alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_5\right),\\
s_1 &: (*) \to \left(x, y - \frac{\alpha_1}{x-z}, z, w + \frac{\alpha_1}{x-z}, q, p, t; \alpha_0 + \alpha_1, -\alpha_1, \alpha_2 + \alpha_1, \alpha_3, \alpha_4, \alpha_5\right),\\
s_2 &: (*) \to \left(x, y, z + \frac{\alpha_2}{w}, w, q, p, t; \alpha_0, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4, \alpha_5\right),\\
s_3 &: (*) \to \left(x, y, z, w - \frac{\alpha_3 q}{zq-1}, q, p - \frac{\alpha_3 z}{zq-1}, t; \alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4 + \alpha_3, \alpha_5 + \alpha_3\right),\\
s_4 &: (*) \to \left(x, y, z, w, q + \frac{\alpha_4}{p}, p, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3 + \alpha_4, -\alpha_4, \alpha_5\right),\\
s_5 &: (*) \to \left(x, y, z, w, q + \frac{\alpha_5}{p-t}, p, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3 + \alpha_5, \alpha_4, -\alpha_5\right),\\
\pi &: (*) \to (x, y, z, w, q, p - t, -t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_4).
\end{aligned}
$$

PROPOSITION 3.5. *Let us define the following translation operators:*

$$
\begin{aligned}
&T_1 := \pi s_4 s_3 s_2 s_1 s_0 s_1 s_2 s_3 s_4, \quad T_2 := \pi s_5 s_4 s_3 s_2 s_1 s_0 s_1 s_2 s_3,\\
&T_3 := s_3 s_5 T_1 s_5 s_3, \quad T_4 := s_2 T_3 s_2, \quad T_5 := s_1 T_4 s_1.
\end{aligned}
\tag{35}
$$

These translation operators act on parameters α_i *as follows:*

$$
\begin{aligned}
T_1(\alpha_0, \alpha_1, \dots, \alpha_5) &= (\alpha_0, \alpha_1, \dots, \alpha_5) + (0, 0, 0, 0, -1, 1),\\
T_2(\alpha_0, \alpha_1, \dots, \alpha_5) &= (\alpha_0, \alpha_1, \dots, \alpha_5) + (0, 0, 0, -1, 1, 1),\\
T_3(\alpha_0, \alpha_1, \dots, \alpha_5) &= (\alpha_0, \alpha_1, \dots, \alpha_5) + (0, 0, 1, -1, 0, 0),\\
T_4(\alpha_0, \alpha_1, \dots, \alpha_5) &= (\alpha_0, \alpha_1, \dots, \alpha_5) + (0, 1, -1, 0, 0, 0),\\
T_5(\alpha_0, \alpha_1, \dots, \alpha_5) &= (\alpha_0, \alpha_1, \dots, \alpha_5) + (1, -1, 0, 0, 0, 0).
\end{aligned}
\tag{36}
$$

THEOREM 3.6. *Let us consider a polynomial Hamiltonian system with Hamiltonian* $H \in \mathbb{C}(t)[x, y, z, w, q, p]$. *We assume that*

$(B1)$ *deg*$(H) = 4$ *with respect to* x, y, z, w, q, p.

$(B2)$ *This system becomes again a polynomial Hamiltonian system in each coordinate system* r_i $(i = 0, 1, 2, 4, 5)$:

$$
\begin{aligned}
&r_0 : x_0 = x + \frac{2\alpha_0}{y} + \frac{1}{y^2},\ y_0 = y,\ z_0 = z,\ w_0 = w,\ q_0 = q,\ p_0 = p,\\
&r_1 : x_1 = -((x - z)y - \alpha_1)y,\ y_1 = 1/y,\ z_1 = z,\ w_1 = w + y,\ q_1 = q,\ p_1 = p,\\
&r_2 : x_2 = x,\ y_2 = y,\ z_2 = 1/z,\ w_2 = -(wz + \alpha_2)z,\ q_2 = q,\ p_2 = p,\\
&r_4 : x_4 = x,\ y_4 = y,\ z_4 = z,\ w_4 = w,\ q_4 = 1/q,\ p_4 = -(pq + \alpha_4)q,\\
&r_5 : x_5 = x,\ y_5 = y,\ z_5 = z,\ w_5 = w.\ q_5 = 1/q,\ p_5 = -((p - t)q + \alpha_5)q.
\end{aligned}
$$

$(B3)$ *In addition to the assumption* $(B2)$*, the Hamiltonian system in the coordinate system* $(x_4, y_4, z_4, w_4, q_4, p_4)$ *becomes again a polynomial Hamiltonian system in the coordinate system* r_3:

$$r_3 : x_3 = x_4,\ y_3 = y_4,\ z_3 = -((z_4 - q_4)w_4 - \alpha_3)w_4,\ w_3 = 1/w_4,\ q_3 = q_4,\ p_3 = p_4 + w_4.$$

Then such a system coincides with the system (17) with the polynomial Hamiltonian (18).

We note that the conditions $(B2)$ and $(B3)$ should be read that

$$r_j(H) \quad (j = 0, 1, 2, 4), \quad r_5(H + q), \quad r_3(r_4(H))$$

are polynomials with respect to x, y, z, w, q, p or $x_4, y_4, z_4, w_4, q_4, p_4$.

Theorems 3.4 and 3.6 can be checked by a direct calculation, respectively.

Next, we show the confluence process from the system (10) to the system (17).

THEOREM 3.7. *For the system* (10) *of type* $D_6^{(1)}$*, we make the change of parameters and variables*

(37)
$$\alpha_0 = \frac{1}{\varepsilon} + 2A_0, \quad \alpha_1 = -\frac{1}{\varepsilon}, \quad \alpha_2 = A_1, \quad \alpha_3 = A_2, \quad \alpha_4 = A_3, \quad \alpha_5 = A_4, \quad \alpha_6 = A_5,$$

(38)
$$t = \varepsilon T, \quad x = \varepsilon X, \quad y = \frac{Y}{\varepsilon}, \quad z = \varepsilon Z, \quad w = \frac{W}{\varepsilon}, \quad q = \frac{Q}{\varepsilon}, \quad p = \varepsilon P$$

from $\alpha_0, \alpha_1, \dots, \alpha_6, x, y, z, w, q, p$ *to* $A_0, A_1, \dots, A_5, X, Y, Z, W, Q, P$*. Then the system* (10) *can also be written in the new variables* X, Y, Z, W, Q, P *and parameters* $A_0, A_1, \dots, A_5$ *as a Hamiltonian system. This new system tends to the system* (17) *with the Hamiltonian* (18) *as* $\varepsilon \to 0$.

Finally, we show the relation between the system (17) and the system of type $D_5^{(1)}$ (see [15]).

THEOREM 3.8. *For the system* (17) *of type* $B_5^{(1)}$*, we make the change of parameters and variables*

(39)
$$\alpha_0 = \frac{A_0 - A_1}{2}, \quad \alpha_1 = A_1, \quad \alpha_2 = A_2, \quad \alpha_3 = A_3, \quad \alpha_4 = A_4, \quad \alpha_5 = A_5,$$

(40)
$$X = ((x - z)y - \alpha_1)y, \quad Y = -1/y, \quad Z = z, \quad W = w + y, \quad Q = q, \quad P = p$$

from $\alpha_0, \alpha_1, \dots, \alpha_5, x, y, z, w, q, p$ *to* $A_0, A_1, \dots, A_5, X, Y, Z, W, Q, P$*. Then the system* (10) *can also be written in the new variables* X, Y, Z, W, Q, P *and parameters* $A_0, A_1, \dots, A_5$ *as a Hamiltonian system. This new system tends to*

(41)
$$\frac{dx}{dt} = \frac{\partial H}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial H}{\partial x}, \quad \frac{dz}{dt} = \frac{\partial H}{\partial w}, \quad \frac{dw}{dt} = -\frac{\partial H}{\partial z}, \quad \frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

with the polynomial Hamiltonian

$$
\begin{aligned}
H =&\frac{x^2y^2+xy^2-(\alpha_0+\alpha_1)xy-\alpha_0y}{t}+\frac{z^2w^2+(\alpha_0+\alpha_1+2\alpha_2)zw+z+tw}{t}\\
&+\frac{q^2p^2-tq^2p-(1-\alpha_4-\alpha_5)qp-\alpha_4tq}{t}+\frac{2(xz-wp)}{t}\\
&(\alpha_0+\alpha_1+2\alpha_2+2\alpha_3+\alpha_4+\alpha_5=1).
\end{aligned}
\tag{42}
$$

Here, for notational convenience, we have renamed A_i, X, Y, Z, W, Q, P to $\alpha_i, x, y, z, w, q, p$ (which are not the same as the previous $\alpha_i, x, y, z, w, q, p$).

We recall that this system admits extended affine Weyl group symmetry of type $D_5^{(1)}$ (see [15]) as the group of its Bäcklund transformations whose generators are explicitly given as follows: with the notation $(*) := (x, y, z, w, q, p, t; \alpha_0, \alpha_1, \dots, \alpha_5)$,

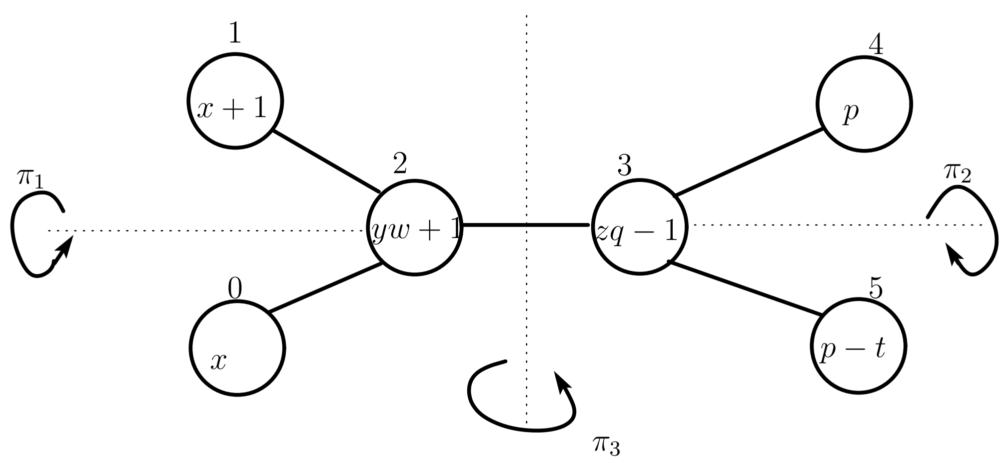

FIGURE 2. Dynkin diagram of type $D_5^{(1)}$

$$
\begin{aligned}
&s_0 : (*) \to \left(x, y-\frac{\alpha_0}{x}, z, w, q, p, t; -\alpha_0, \alpha_1, \alpha_2+\alpha_0, \alpha_3, \alpha_4, \alpha_5\right),\\
&s_1 : (*) \to \left(x, y-\frac{\alpha_1}{x+1}, z, w, q, p, t; \alpha_0, -\alpha_1, \alpha_2+\alpha_1, \alpha_3, \alpha_4, \alpha_5\right),\\
&s_2 : (*) \to \left(x+\frac{\alpha_2 w}{yw+1}, y, z+\frac{\alpha_2 y}{yw+1}, w, q, p, t; \alpha_0+\alpha_2, \alpha_1+\alpha_2, -\alpha_2, \alpha_3+\alpha_2, \alpha_4, \alpha_5\right),\\
&s_3 : (*) \to \left(x, y, z, w-\frac{\alpha_3 q}{zq-1}, q, p-\frac{\alpha_3 z}{zq-1}, t; \alpha_0, \alpha_1, \alpha_2+\alpha_3, -\alpha_3, \alpha_4+\alpha_3, \alpha_5+\alpha_3\right),\\
&s_4 : (*) \to \left(x, y, z, w, q+\frac{\alpha_4}{p}, p, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3+\alpha_4, -\alpha_4, \alpha_5\right),\\
&s_5 : (*) \to \left(x, y, z, w, q+\frac{\alpha_5}{p-t}, p, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3+\alpha_5, \alpha_4, -\alpha_5\right),
\end{aligned}
$$

$$\pi_1 : (*) \rightarrow (-x-1, -y, -z, -w, -q, -p, -t; \alpha_1, \alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_5),$$
$$\pi_2 : (*) \rightarrow (x, y, z, w, q, p-t, -t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_4),$$
$$\pi_3 : (*) \rightarrow \left(\frac{(p-t)}{t}, -tq, -tw, \frac{z}{t}, \frac{y}{t}, -t(x+1), -t; \alpha_5, \alpha_4, \alpha_3, \alpha_2, \alpha_1, \alpha_0\right).$$

4. The system of type $D_5^{(2)}$

In this section, we present a 4-parameter family of coupled Hamiltonian systems in dimension six with extended affine Weyl group symmetry of type $D_5^{(2)}$ given by

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{\partial H}{\partial y} = \dfrac{x^2y + z^2w + \alpha_0 x + \alpha_2 z - p}{t}, \\ \dfrac{dy}{dt} = -\dfrac{\partial H}{\partial x} = \dfrac{-2xy^2 - 2\alpha_0 y - 1}{2t}, \\ \dfrac{dz}{dt} = \dfrac{\partial H}{\partial w} = \dfrac{z^2w + yz^2 + (\alpha_0 + \alpha_1 + \alpha_2)z - p}{t}, \\ \dfrac{dw}{dt} = -\dfrac{\partial H}{\partial z} = \dfrac{-zw^2 - 2yzw - \alpha_2 y - (\alpha_0 + \alpha_1 + \alpha_2)w}{t}, \\ \dfrac{dq}{dt} = \dfrac{\partial H}{\partial p} = \dfrac{q^2p - y - w + (\alpha_4 - 1)q}{t}, \\ \dfrac{dp}{dt} = -\dfrac{\partial H}{\partial q} = \dfrac{-2qp^2 - 2(\alpha_4 - 1)p + t}{2t} \end{cases} \tag{43}$$

with the polynomial Hamiltonian

$$\begin{aligned} H &= \frac{x^2y^2 + 2\alpha_0 xy + x}{2t} + \frac{z^2w^2 + 2(\alpha_0 + \alpha_1 + \alpha_2)zw}{2t} \\ &+ \frac{q^2p^2 + 2(\alpha_4 - 1)qp - tq}{2t} + \frac{yz(zw + \alpha_2)}{t} - \frac{(y+w)p}{t} \\ &= H_3(x, y, t; 2\alpha_0) + H_4(z, w, t; 2(\alpha_0 + \alpha_1 + \alpha_2)) \\ &+ H_5(q, p, t; 2(\alpha_4 - 1)) + \frac{yz(zw + \alpha_2)}{t} - \frac{(y+w)p}{t}. \end{aligned} \tag{44}$$

Here x, y, z, w, q and p denote unknown complex variables and $\alpha_0, \alpha_1, \dots, \alpha_4$ are complex parameters satisfying the relation:

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1. \tag{45}$$

The symbols H_3, H_4 and H_5 are given by

$$H_3(q, p, t; \alpha) = \frac{q^2p^2 + \alpha qp + q}{2t}, \tag{46}$$

$$H_4(q, p, t; \alpha) = \frac{q^2p^2 + \alpha qp}{2t}, \tag{47}$$

$$H_5(q, p, t; \alpha, \beta) = \frac{q^2p^2 + \alpha qp - tq}{2t}. \tag{48}$$

PROPOSITION 4.1. *The system with the Hamiltonian* H_3

$$\frac{dq}{dt}=\frac{\partial H_3}{\partial p},\quad \frac{dp}{dt}=-\frac{\partial H_3}{\partial q} \tag{49}$$

has the first integral I_3:

$$I_3=q^2p^2+\alpha qp+q. \tag{50}$$

PROPOSITION 4.2. *The system with the Hamiltonian* H_4

$$\frac{dq}{dt}=\frac{\partial H_4}{\partial p},\quad \frac{dp}{dt}=-\frac{\partial H_4}{\partial q} \tag{51}$$

has the first integral I_4:

$$I_4=qp. \tag{52}$$

PROPOSITION 4.3. *By the following birational and symplectic transformation* tr_5:

$$tr_5(q,p)=(tq,p/t), \tag{53}$$

the Hamiltonians H_5 *satisfy the following relation*:

$$tr_5(H_5)=\frac{q^2p^2+\alpha qp-q}{2t}. \tag{54}$$

By Proposition 4.3, we see that the Hamiltonian system with

$$K_5:=\frac{q^2p^2+\alpha qp-q}{2t}$$

has the first integral.

PROPOSITION 4.4. *The system with the Hamiltonian* K_5

$$\frac{dq}{dt}=\frac{\partial K_5}{\partial p},\quad \frac{dp}{dt}=-\frac{\partial K_5}{\partial q} \tag{55}$$

has the first integral I_5:

$$I_5=q^2p^2+\alpha qp-q. \tag{56}$$

In this case, each differential system with respect to all principal parts H_3, H_4 and H_5 has its first integral. Nevertheless, the Hamiltonian H is not the first integral. For this system we tried to seek its first integrals of polynomial type with respect to x, y, z, w, q, p. However, we can not find.

THEOREM 4.5. *The system* (43) *admits extended affine Weyl group symmetry of type* $D_5^{(2)}$ *as the group of its Bäcklund transformations, whose generators are explicitly given*

as follows: *with the notation* $(*) := (x, y, z, w, q, p, t; \alpha_0, \alpha_1, \dots, \alpha_4)$,

$$\begin{aligned}
&s_0 : (*) \rightarrow \left(-x - \frac{2\alpha_0}{y} - \frac{1}{y^2}, -y, -z, -w, -q, -p, -t; -\alpha_0, \alpha_1 + 2\alpha_0, \alpha_2, \alpha_3, \alpha_4\right),\\
&s_1 : (*) \rightarrow \left(x, y - \frac{\alpha_1}{x-z}, z, w + \frac{\alpha_1}{x-z}, q, p, t; \alpha_0 + \alpha_1, -\alpha_1, \alpha_2 + \alpha_1, \alpha_3, \alpha_4\right),\\
&s_2 : (*) \rightarrow \left(x, y, z + \frac{\alpha_2}{w}, w, q, p, t; \alpha_0, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4\right),\\
&s_3 : (*) \rightarrow \left(x, y, z, w - \frac{\alpha_3 q}{zq-1}, q, p - \frac{\alpha_3 z}{zq-1}, t; \alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4 + \alpha_3\right),\\
&s_4 : (*) \rightarrow \left(x, y, z, w, q + \frac{2\alpha_4}{p} - \frac{t}{p^2}, p, -t; \alpha_0, \alpha_1, \alpha_2, \alpha_3 + 2\alpha_4, -\alpha_4\right),\\
&\pi : (*) \rightarrow \left(-tq, -\frac{p}{t}, -\frac{t}{z}, \frac{(zw + \alpha_2)z}{t}, -\frac{x}{t}, -ty, t; \alpha_4, \alpha_3, \alpha_2, \alpha_1, \alpha_0\right).
\end{aligned}$$

PROPOSITION 4.6. *Let us define the following translation operators*:

$$\begin{aligned}
&T_1 := s_4 s_3 s_2 s_1 s_0 s_1 s_2 s_3, \quad T_2 := s_3 T_1 s_3,\\
&T_3 := s_2 T_2 s_2, \quad T_4 := s_1 T_3 s_1.
\end{aligned} \tag{57}$$

These translation operators act on parameters α_i *as follows*:

$$\begin{aligned}
T_1(\alpha_0, \alpha_1, \dots, \alpha_4) =&(\alpha_0, \alpha_1, \dots, \alpha_4) + (0, 0, 0, -2, 2),\\
T_2(\alpha_0, \alpha_1, \dots, \alpha_4) =&(\alpha_0, \alpha_1, \dots, \alpha_4) + (0, 0, -2, 2, 0),\\
T_3(\alpha_0, \alpha_1, \dots, \alpha_4) =&(\alpha_0, \alpha_1, \dots, \alpha_4) + (0, -2, 2, 0, 0),\\
T_4(\alpha_0, \alpha_1, \dots, \alpha_4) =&(\alpha_0, \alpha_1, \dots, \alpha_4) + (-2, 2, 0, 0, 0).
\end{aligned} \tag{58}$$

THEOREM 4.7. *Let us consider a polynomial Hamiltonian system with Hamiltonian* $H \in \mathbb{C}(t)[x, y, z, w, q, p]$. *We assume that*

$(C1)$ *deg*$(H) = 4$ *with respect to* x, y, z, w, q, p.

$(C2)$ *This system becomes again a polynomial Hamiltonian system in each coordinate system* r_i $(i = 0, 1, 2, 4)$:

$$\begin{aligned}
&r_0 : x_0 = x + \frac{2\alpha_0}{y} + \frac{1}{y^2},\ y_0 = y,\ z_0 = z,\ w_0 = w,\ q_0 = q,\ p_0 = p,\\
&r_1 : x_1 = -((x - z)y - \alpha_1)y,\ y_1 = 1/y,\ z_1 = z,\ w_1 = w + y,\ q_1 = q,\ p_1 = p,\\
&r_2 : x_2 = x,\ y_2 = y,\ z_2 = 1/z,\ w_2 = -(wz + \alpha_2)z,\ q_2 = q,\ p_2 = p,\\
&r_4 : x_4 = x,\ y_4 = y,\ z_4 = z,\ w_4 = w.\ q_4 = q + \frac{2\alpha_4}{p} - \frac{t}{p^2},\ p_4 = p.
\end{aligned}$$

$(C3)$ *In addition to the assumption* $(C2)$, *the Hamiltonian system in the coordinate system* $(x_2, y_2, z_2, w_2, q_2, p_2)$ *becomes again a polynomial Hamiltonian system in the coordinate system* r_3:

$$r_3 : x_3 = x_2,\ y_3 = y_2,\ z_3 = -((z_2 - q_2)w_2 - \alpha_3)w_2,\ w_3 = 1/w_2,\ q_3 = q_2,\ p_3 = p_2 + w_2.$$

Then such a system coincides with the system (43) with the polynomial Hamiltonian (44).

We note that the conditions $(C2)$ and $(C3)$ should be read that

$$r_j(H) \quad (j=0,1,2), \quad r_4(H+1/p), \quad r_3(r_2(H))$$

are polynomials with respect to x,y,z,w,q,p or x_2,y_2,z_2,w_2,q_2,p_2.

Theorems 4.5 and 4.7 can be checked by a direct calculation, respectively.

Next, we show the confluence process from the system (10) to the system (43).

Theorem 4.8. *For the system* (10) *of type* $D_6^{(1)}$, *we make the change of parameters and variables*

(59)

$$\alpha_0=-\frac{1}{\varepsilon}+A_0, \quad \alpha_1=\frac{1}{\varepsilon}, \quad \alpha_2=\frac{A_1}{2}, \quad \alpha_3=\frac{A_2}{2}, \quad \alpha_4=\frac{A_3}{2}, \quad \alpha_5=\frac{1}{\varepsilon}, \quad \alpha_6=-\frac{1}{\varepsilon}+A_4,$$

$$t=\frac{\varepsilon^2}{16}T, \quad x=\frac{\varepsilon}{4}X, \quad y=\frac{2Y}{\varepsilon}, \quad z=\frac{\varepsilon}{4}Z, \quad w=\frac{2W}{\varepsilon}, \quad q=\frac{4Q}{\varepsilon}, \quad p=\frac{\varepsilon}{8}P \tag{60}$$

from $\alpha_0,\alpha_1,\dots,\alpha_6,x,y,z,w,q,p$ *to* $A_0,A_1,\dots,A_4,X,Y,Z,W,Q,P$. *Then the system* (10) *can also be written in the new variables* X,Y,Z,W,Q,P *and parameters* $A_0,A_1,\dots,A_4$ *as a Hamiltonian system. This new system tends to the system* (43) *with the Hamiltonian* (44) *as* $\varepsilon\to 0$.

Finally, we find an algebraic solution of the system (43):

$$\begin{aligned}&(\alpha_0,\alpha_1,\dots,\alpha_4)=(\alpha_0,\alpha_1,1-2\alpha_0-2\alpha_1,\alpha_1,\alpha_0)\,,\\ &(x,y,z,w,q,p)=(-\left(\frac{1+\sqrt{-1}}{4}\right)(t^{\frac14}+2(1+\sqrt{-1})\sqrt{t}),-\frac{1-\sqrt{-1}}{2t^{\frac14}},\sqrt{-t},\\ &-\frac{\sqrt{-1}(2\alpha_0+2\alpha_1-1)}{2\sqrt{t}},\frac{(1+\sqrt{-1})+4\sqrt{-1}t^{\frac14}}{4t^{\frac34}},\left(\frac12-\frac{\sqrt{-1}}{2}\right)t^{\frac34}).\end{aligned} \tag{61}$$

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